CDI-II - Prática F.4 24/3/21

Ficha 4

Derivada da função composta

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "g", "\text{dif. em } a"'] & \mathbb{R}^p \arrow[r, "f", "\text{dif. em } g(a)"'] & \mathbb{R}^m \end{tikzcd}$$

$$\begin{tikzcd} a \arrow[r, mapsto] \arrow[rr, bend right, "f\circ g \text{ dif. em } a"'] & g(a) \arrow[r, mapsto] & f(g(a)) \end{tikzcd}$$

$$\underbrace{D(f\circ g)(a)}_{m\times n} = \underbrace{Df(g(a))}_{m\times p}\,\underbrace{Dg(a)}_{p\times n}$$

1 — Trivial : $g(1,1) = (1,0)$

$$D(f\circ g)(1,1) = Df(1,0)\, Dg(1,1)$$

$$\boxed{(1,1)} \longrightarrow \underset{\boxed{(1,0)}}{\underset{\parallel}{g(1,1)}} \longrightarrow f(g(1,1))$$

— " —

2 - $\gamma \equiv$ "gama" , $\sigma \equiv$ "sigma"

$$\sigma(t) = F(\gamma(t))$$

$$\mathbb{R} \xrightarrow{\gamma} \mathbb{R}^3 \xrightarrow{F} \mathbb{R}$$

$$t \longmapsto \gamma(t) \longmapsto F(\gamma(t))$$

$$\underset{1\times 1}{\sigma'(t)} = \underset{1\times 3}{DF(\gamma(t))}\, \underset{3\times 1}{\gamma'(t)}$$

$$\sigma'(t) = \underbrace{\begin{bmatrix} 2\sin t & 2t^2 & 2\cos t \end{bmatrix}}_{DF(\gamma(t))} \begin{bmatrix} \cos t \\ 2t \\ -\sin t \end{bmatrix} = 4t^3$$

$$DF(x,y,z) = \begin{bmatrix} 2x & 2y & 2z \end{bmatrix}$$

$$DF(\gamma(t)) = \begin{bmatrix} 2\sin t & 2t^2 & 2\cos t \end{bmatrix}.$$

$$\gamma(t) = (\underbrace{\sin t}_{x(t)}, \underbrace{t^2}_{y(t)}, \underbrace{\cos t}_{z(t)})$$

$$\gamma'(t) = \begin{bmatrix} \cos t \\ 2t \\ -\sin t \end{bmatrix}$$

$$\sigma'(t) = 4t^3.$$

Alternativamente, usar a regra da cadeia.

$$\sigma(t) = F(\gamma(t)) = F(x(t), y(t), z(t))$$

$2x = 2\,\mathrm{sen}\,t$

$$\sigma'(t) = \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt}$$

$x(t) = \mathrm{sen}\,t$
$y(t) = t^2$
$z(t) = \cos t$

$(x(t), y(t), z(t))$ $\quad$ (t)

$$= \cancel{2\,\mathrm{sen}\,t\cos t} + 2t^2 \times 2t + \cancel{2\cos t(-\mathrm{sen}\,t)}$$

$$= 4t^3.$$

3 -

$$\mathbb{R}^2 \xrightarrow{g} \mathbb{R}^3 \xrightarrow{f} \mathbb{R}$$

$$(0,0) \longmapsto g(0,0) \longmapsto f(g(0,0)) = f(0,1,2)$$

$$g(0,0) = (0,1,2)$$

$$D(f\circ g)(0,0) = Df(\underbrace{0,1,2}_{g(0,0)})\, Dg(0,0)$$

$$= \begin{bmatrix} \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}$$

$$D_v(f\circ g)(0,0) = D(f\circ g)(0,0)\, v$$

$$\begin{bmatrix} \cdot & \cdot \end{bmatrix} \begin{bmatrix} \cdot \\ \cdot \end{bmatrix}$$

etc.

Alternativamente, usar a regra de cadeia.

$$\underbrace{(f\circ g)}_{h}(x,y) = f\big(g(x,y)\big)$$

$$h(x,y) = f\big(u(x,y), v(x,y), w(x,y)\big)$$

$$f(u,v,w) = v e^{u} + u w^{2}$$

$$\frac{\partial h}{\partial x}(x,y) = \frac{\partial f}{\partial u}\,\overset{=0}{\frac{\partial u}{\partial x}} + \frac{\partial f}{\partial v}\,\overset{=2}{\frac{\partial v}{\partial x}} + \frac{\partial f}{\partial w}\,\overset{=4}{\frac{\partial w}{\partial x}}$$

etc.

$(0,1,2)$ $\quad$ $(0,0)$

(u,v,w) $\quad$ (x,y)

$\parallel$

$g(x,y)$

4 — $\sigma(x) = f(g(x))$

$g(x) = (\operatorname{sen} x, x + e^{x})$

$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$g : \mathbb{R} \to \mathbb{R}^2$

$$\begin{tikzcd}
\mathbb{R} \arrow[r, "g"] & \mathbb{R}^2 \arrow[r, "f"] & \mathbb{R}^3 \\
x \arrow[r, mapsto] \arrow[rr, bend right, "\sigma(x) = (f\circ g)(x)"'] & g(x) = (\operatorname{sen} x, x+e^{x}) \arrow[r, mapsto] & f(g(x))
\end{tikzcd}$$

$\sigma'(x) = Df(g(x))\, Dg(x)$

$\sigma'(0) = Df(g(0))\, Dg(0)$

$$\sigma'(0) = \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{bmatrix} \begin{bmatrix} \cdot \\ \cdot \end{bmatrix} \quad \text{etc.}$$

5-a)

$$\begin{tikzcd} \mathbb{R}^3 \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & \mathbb{R}^3 \arrow[r, "g"] & \mathbb{R} \end{tikzcd}$$

$$(x,y,z) \longmapsto f(x,y,z) \longrightarrow g(f(x,y,z))$$

$$(1,1,0) \longmapsto (2,2,1) \longmapsto g(2,2,1)$$

$$D(g\circ f)\underset{(1,1,0)}{(x,y,z)} = Dg\Big(\underset{(2,2,1)}{f(x,y,z)}\Big)\, Df\underset{(1,1,0)}{(x,y,z)}$$

$$= \begin{bmatrix} \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}$$

$$\frac{\partial}{\partial y}(g\circ f)(1,1,0)$$

etc.

c/ Règle de chaîne :

$$(g \circ f)(x,y,z) = h(x,y,z)$$

$$h(x,y,z) = g\big(f(x,y,z)\big) = g\big(\underbrace{u(x,y,z), v(x,y,z), w(x,y,z)}_{f(x,y,z)}\big)$$

$$u(x,y,z) = x^2 + y^2 + z^2$$

$$v(x,y,z) = x + y - z$$

$$w(x,y,z) = xy e^{z}$$

$$\nabla g(2,2,1) = (-1,0,3)$$

$$\frac{\partial h}{\partial y} = \overset{-1}{\frac{\partial g}{\partial u}} \frac{\partial u}{\partial y} + \overset{0}{\frac{\partial g}{\partial v}} \frac{\partial v}{\partial y} + \overset{3}{\frac{\partial g}{\partial w}} \frac{\partial w}{\partial y}$$

(x,y,z)

$(1,1,0)$

(u,v,w)

$(2,2,1)$

(x,y,z)

5-b) $\frac{\partial}{\partial z}(g \circ f)(0,1,0)$

$$h(x,y,z) = g(f(x,y,z))$$

$$h(x,y,z) = g(u(x,y,z), v(x,y,z), w(x,y,z))$$

$$\frac{\partial h}{\partial z} = \frac{\partial g}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial g}{\partial v}\frac{\partial v}{\partial z} + \frac{\partial g}{\partial w}\frac{\partial w}{\partial z}$$

(x,y,z)

$(0,1,0)$

$$f(x,y,z) = (u,v,w)$$

$$f(0,1,0) = (1,1,0)$$

$(0,1,0)$

etc.

7- $F(x, y, g(x, y)) = 0$ ✓

$\Longleftrightarrow$

$z = g(x, y) \equiv z(x, y)$

$\boxed{\frac{\partial F}{\partial z}(x, y, z) \neq 0} \Longrightarrow Dg(x, y)?$

$\underbrace{F(x, y, z(x, y))}_{h(x,y)} = 0$

$h(x, y) = 0$

$\frac{\partial g}{\partial x} = \frac{\partial z}{\partial x}$

$\frac{\partial g}{\partial y} = \frac{\partial z}{\partial y}$

$0 = \frac{\partial h}{\partial x}(x, y) = \frac{\partial F}{\partial x} + \underbrace{\frac{\partial F}{\partial z}}_{\neq 0} \boxed{\frac{\partial z}{\partial x}}$

$0 = \frac{\partial h}{\partial y}(x, y) = \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z} \boxed{\frac{\partial z}{\partial y}}$

$$\frac{\partial g}{\partial x}(x,y) = - \frac{\frac{\partial F}{\partial x}(x,y,g(x,y))}{\frac{\partial F}{\partial z}(x,y,g(x,y))}$$

$$\frac{\partial g}{\partial y}(x,y) = - \frac{\frac{\partial F}{\partial y}(x,y,g(x,y))}{\frac{\partial F}{\partial z}(x,y,g(x,y))}$$

$$6 - \frac{\partial}{\partial x} g\left(g(x^2, xy, x+y), 3xy, 2x+y\right)$$

$$g\left(\overbrace{g(x^2, xy, x+y)}^{u(x,y)}, \overbrace{3xy}^{v(x,y)}, \overbrace{2x+y}^{w(x,y)}\right)$$

$$= g\left(u(x,y), v(x,y), w(x,y)\right)$$

$$h(x,y) = g\big(u(x,y), v(x,y), w(x,y)\big)$$

$$\begin{cases} u(x,y) = g(x^2, xy, x+y) \\ v(x,y) = 3xy \\ w(x,y) = 2x+y \end{cases}$$

$$\frac{\partial h}{\partial x} = \frac{\partial g}{\partial u}\boxed{\frac{\partial u}{\partial x}} + \frac{\partial g}{\partial v}\overset{3y}{\frac{\partial v}{\partial x}} + \frac{\partial g}{\partial w}\overset{2}{\frac{\partial w}{\partial x}}$$

$$u(x,y) = g\big(a(x,y), b(x,y), c(x,y)\big)$$

$$\frac{\partial u}{\partial x} = \frac{\partial g}{\partial a}\frac{\partial a}{\partial x} + \frac{\partial g}{\partial b}\frac{\partial b}{\partial x} + \frac{\partial g}{\partial c}\frac{\partial c}{\partial x}$$

$\dfrac{\partial g}{\partial u}$, $\dfrac{\partial g}{\partial a}$ = derivada parcial de g na primeira variável

(u,v,w) $\quad$ (a,b,c)